(T-954) Roll No. .....

ID—201-CU-305612

M. Sc. EXAMINATION, 2023

(Third Semester)

(w.e.f 2020 UTD's)

MATHEMATICS

Code : MATH-231

Analytical Number Theory

Time : 3 Hours *Maximum Marks : 80*

Before answering the question-paper candidates should ensure that they have been supplied to correct and complete question-paper. No complaint, in this regard, will be entertained after the examination.

Note : This question paper consists of **five Sections.** Question No. **1** in Section I is **compulsory,** consisting of eight short answer type

questions each carrying 2 marks. Each of the last four Sections (II-V) contains two questions each carrying 16 marks. Attempt *one* question from each Section.

Section I

1. (a) What does Euclid's Theorem tell us about prime numbers ?

   (b) What are Fermat Numbers ? Give an example.

   (c) Find the units in Z_8.

   (d) Find the number of elements of order 4 in U_{10}.

   (e) What is completely multiplicative function ? Give an example.

   (f) What is Diophantine equation ?

   (g) Find the value of $\phi(12)$.

   (h) Find the value of $\tau(21)$. 8×2=16

Section II

2. (a) Let $a \geq 2$, $n \geq 2$ be natural numbers. If $a^n - 1$ is a prime, then show that $a = 2$ and $n = p$ for some prime number p. 8

   (b) Show that F_5 is a composite number. 8

3. (a) Prove that π is irrational. 8

   (b) Given any irrational number ξ, show that there exist infinitely many pairs (h, k) of integers such that $\left|\xi - \frac{h}{k}\right| < \frac{1}{\sqrt{5}k^2}$. 8

Section III

4. (a) Solve the simultaneous congruences :

   $x \equiv 2 \bmod(7)$, $x \equiv 7 \bmod(9)$,

   $x \equiv 3 \bmod(4)$. 8

   (b) Find all the solutions of $x^2 + y^2 = z^2$, where $0 < z < 24$. 8

5. (a) Prove that every natural number n can be written as a sum of four squares. 8

   (b) Let p be a prime such that $p > 2$ (that is, p is an odd prime), then show that $G\left[p^\theta(p-1)\right] \geq p^{\theta+1}$. 8

Section IV

6. (a) Let p be an odd prime. Then, the quadratic congruence $x^2 + 1 \equiv 0 \pmod{p}$ has a solution if and only if $p \equiv 1 \pmod{4}$. 8

   (b) Prove that the Group $U_2 i$ is cyclic if and only if $i = 1$ or $i = 2$. 8

7. (a) Find the order of elements of U_{10}. 8

   (b) Find primitive roots in U_n for $n = 28$. 8

Section V

8. (a) Prove that the Mobius function $\mu(n)$ is a multiplicative function. 8

(b) Show that :

$$\phi(n)=\sum_{\frac{d}{n}}\mu(d)\left(\frac{n}{d}\right)=-\sum_{\frac{d}{n}}d\,\mu\left(\frac{n}{d}\right).$$ **8**

9. (a) Let $n > 1$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots\ldots, p_k$ are distinct primes. Then prove that :

$$\phi(n)=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_k}\right).$$ **8**

(b) Prove that $\sigma(n)$ is odd $\Leftrightarrow n=m^2$ or $n=2m^2$. **8**